II. Note on Professor Bennett's solution by H. P. Manning, Providence, R. I., and Otto Dunkel, Washington University.

If $f(x)$ can be expanded to four terms with the Lagrange remainder, the substitution of this expansion will show that $\max |g(x)| \leqq \frac{h^5}{3!} \max |f^{iv}(x)|$, but substitution in the integral $\int_{a-h}^{a+h} g(x)dx$ gives for the error the smaller limit $\frac{2}{45} h^5 \max |f^{iv}(x)|$, the same that is given by the substitution for $f(x)$, and for $f(a+h)$ and $f(a-h)$ in Simpson's formula itself. This limit is given in E. B. Wilson's *Advanced Calculus*, 1912, pages 76–77, exercises 23 and 24. However, a limit still smaller, namely, $\frac{h^5}{90} \max |f^{iv}(x)|$, has been found and is given by P. J. Daniell in this Monthly, *1917*, 110, and also by C. J. de la Vallée-Poussin, *Cours d'Analyse Infinitésimale*, volume 1, third edition, 1914, page 396. The method employed by the latter can be applied to the integral $\int_{a-h}^{a+h} g(x)dx$ and leads to the same result. Thus Professor Bennett's expression, obtained without assuming any expansion, leads to the results already found for functions capable of expansion to four terms and a remainder.

In 1874 Chevilliet (*Comptes Rendus de l'Académie des Sciences,* volume 78, page 1841), by taking the infinite expansion of $f(x)$, shows that the first term of the error is $-\frac{h^5}{90} f^{iv}(a)$, and when h is sufficiently small this approximates to the limit given by de la Vallée-Poussin. This result is obtained in the same way in Kiepert's *Grundriss der Differential- und Integralrechnung*, Teil 2, seventh edition, 1900, pages 335–336.

In Heine's *Handbuch der Kugelfunktionen*, Band 2, Theil 1, 1881, there is an exhaustive discussion of mechanical quadrature, and expressions are obtained for the errors in the Newton-Cotes method and in the method of Gauss. In particular, the fraction $-1/90$ can be obtained by multiplying $1/4!$ by the $-4/15$ given in the table on page 9.

William Hoover gave the reference to Wilson's *Calculus,* and H. E. Jordan to Kiepert's work.

2868 [1920, 482]. Proposed by H. S. Uhler, Yale University.

Let the evolute of a given curve be called the evolute of the first order, let the evolute of the first evolute be called the evolute of the second order, etc. Then, being given the following parametric equations in which a is a constant and γ is the parameter, namely,

$$x = (1 + 2\sin^2\gamma)\cos\gamma - a\sin 2\gamma, \qquad y = 2\sin^3\gamma + a\cos 2\gamma,$$

find: (*a*) the parametric equations of the evolute of order n, both for n even and for n odd;
(*b*) a formula for the total length of the nth evolute;
(*c*) a formula for the total area of the nth evolute;
(*d*) the sum of the lengths of all the evolutes from $n = 1$ to $n = \infty$; and
(*e*) the sum of the areas of all the evolutes from $n = 1$ to $n = \infty$.

Note. The original equations represent the envelope required in problem 2819 (*1920*, 134).

I. Solution by F. L. Wilmer, Omaha, Neb., and H. P. Manning, Providence, R. I.

One may note that the equation of the normal to the given curve can be put into the p-form with p a simple function of the parameter. Then differentiation with respect to the latter must give the equation of a perpendicular meeting this line in the corresponding point of the evolute, and so normal to the latter. In this way are obtained the parametric equations of the evolute, and by repetition those of the nth evolute.

We can write the given equations

$$\begin{aligned} x &= 3\cos\gamma - 2\cos^3\gamma - a\sin 2\gamma, \\ y &= \phantom{3\cos\gamma - {}} 2\sin^3\gamma + a\cos 2\gamma; \end{aligned}$$

and from these it follows that $dy/dx = \tan 2\gamma$.

Now the equation of the normal will be

$$X \cos 2\gamma + Y \sin 2\gamma = x \cos 2\gamma + y \sin 2\gamma = \cos \gamma.$$

Thus for the first evolute we have the equations

$$x_1 \cos 2\gamma + y_1 \sin 2\gamma = \cos \gamma, \qquad x_1 \sin 2\gamma - y_1 \cos 2\gamma = (\sin \gamma)/2;$$

or

$$x_1 = \cos^3 \gamma, \qquad 2y_1 = 3 \sin \gamma - 2 \sin^3 \gamma.$$

Similarly, starting with these equations, we get for the second evolute[1]

$$2(x_2 \sin 2\gamma - y_2 \cos 2\gamma) = \sin \gamma, \qquad 2(x_2 \cos 2\gamma + y_2 \sin 2\gamma) = (\cos \gamma)/2;$$

or

$$2^2 x_2 = 3 \cos \gamma - 2 \cos^2 \gamma, \qquad 2y_2 = \sin^3 \gamma.$$

If we let x_0 and y_0 be what x and y become when a is zero, then $2^2 x_2 = x_0$ and $2^2 y_2 = y_0$. Suppose

$$2^{n-1} x_n = \cos^3 \gamma, \qquad 2^n y_n = 3 \sin \gamma - 2 \sin^3 \gamma; \tag{1}$$

so that $2^{n-1} x_n = x_1$ and $2^{n-1} y_n = y_1$. Then we shall have

$$2^{n+1} x_{n+1} = 3 \cos \gamma - 2 \cos^3 \gamma, \qquad 2^n y_{n+1} = \sin^3 \gamma,$$

or $2^{n+1} x_{n+1} = x_0$ and $2^{n+1} y_{n+1} = y_0$, getting these equations in the same way that the equations for x_2 and y_2 were derived from those for x_1 and y_1.

Again, suppose

$$2^n x_n = 3 \cos \gamma - 2 \cos^3 \gamma, \qquad 2^{n-1} y_n = \sin^3 \gamma; \tag{2}$$

so that $2^n x_n = x_0$ and $2^n y_n = y_0$. Then $2^n x_{n+1} = \cos^3 \gamma$, and $2^{n+1} y_{n+1} = 3 \sin \gamma - 2 \sin^3 \gamma$, or $2^n x_{n+1} = x_1$ and $2^n y_{n+1} = y_1$.

Thus we prove by induction that (2) holds for n even and (1) for n odd.

For lengths we have $ds_0/d\gamma = 3 \sin \gamma$, $2ds_1/d\gamma = 3 \cos \gamma$; hence for n even $2^n ds_n/d\gamma = ds_0/d\gamma = 3 \sin \gamma$, and for n odd $2^n ds_n/d\gamma = 2ds_1/d\gamma = 3 \cos \gamma$. Integrating through 90° and multiplying by 4, we have in all cases for the complete length of the nth evolute $3/2^{n-2}$.

The length of the first evolute is 6 and the sum of the lengths of all of the evolutes is

$$6 + \sum_{n=2}^{\infty} 3/2^{n-2} = 12.$$

When n is even $ds_n/d\gamma$ becomes zero and changes sign for $\gamma = 0$ or π; therefore each of these evolutes has two cusps on the x-axis.

When n is odd $ds_n/d\gamma$ becomes zero and changes sign for $\gamma = \pm \pi/2$, and each of these evolutes has two cusps on the y-axis.

Areas can be obtained by the formula $A = 4 \int_0^1 y dx = -4 \int_0^{\pi/2} y \frac{dx}{d\gamma} d\gamma$.

Now $y_0 dx_0/d\gamma = -2 \sin^3 \gamma (3 \sin \gamma - 6 \cos^2 \gamma \sin \gamma)$; therefore $A_0 = 3\pi$.

Also $2y_1 dx_1/d\gamma = (3 \sin \gamma - 2 \sin^3 \gamma)(-3 \cos^2 \gamma \sin \gamma)$; therefore $A_1 = 3\pi/4$.

Then when n is even $2^{2n} A_n = A_0$, and when n is odd $2^{2(n-1)} A_n = A_1$. Thus $A_n = 3\pi/2^{2n}$ for all values of n.

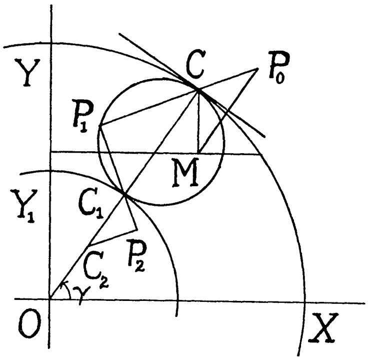
The sum of the areas of all of the evolutes is $\sum_{n=1}^{\infty} 3\pi/2^{2n} = \pi$.

Note. The area between two successive evolutes is swept over by the radius of curvature of the larger as γ varies through 360°. Thus we can write $A_n - A_{n+1} = \int_0^{2\pi} \frac{\rho_n^2}{2} \cdot 2d\gamma$, where ρ_n is the radius of curvature of the nth evolute. But then $\rho_n = s_{n+1} = (1/2^{n+1}) \cdot 3 \sin \gamma$ (or $3 \cos \gamma$ when n is odd). Thus this integral becomes $\frac{9}{2^{2(n+1)}} \int_0^{2\pi} \sin^2 \gamma$ (or $\cos^2 \gamma$) $d\gamma = \frac{9\pi}{2^{2(n+1)}}$.

[1] It is not necessary to go through the process of deriving the first of these two equations, since it is the same as the second of the two equations for x_1 and y_1, the normal to the first evolute being the tangent to the second.

II. Solution by Otto Dunkel, Washington University.

The solution of this problem, like that of 2819 (*1921*, 190) is simplified by the geometry of the curves. These may be constructed as follows: A unit circle is drawn with the origin as center and a chord parallel to the x-axis at a units above it. A radius OC is drawn with the inclination γ to the x-axis and the point C is projected upon the chord in the point M. Then the point P_0 which is symmetrical to M with respect to the tangent at C is a point on the given curve, P_0C is a normal, and the envelope of this normal, the first evolute of the present problem, is the caustic of the unit circle produced by vertical rays. If P_1 is the point of contact with the caustic, $CP_1 = (\sin\gamma)/2$ (formula (1), *1920*, 225, ω of the formula being equal to $\pi/2 - \gamma$ and R to 1), and this point is obtained by projecting the middle point C_1 of OC upon P_0C. P_1P_0 has the inclination $2\gamma - \pi/2$ and length $\rho_0 = (3/2)\sin\gamma - a$. Angle P_1CC_1 = angle YOC_1; therefore the arc P_1C_1 on the circle of diameter C_1C is equal to the arc Y_1C_1, where Y_1 is the point where the y-axis is cut by the circle with center O and radius $OC_1 = 1/2$. It follows that the locus of P_1 is the curve traced by this point when the former circle rolls on the latter circle; it is an epicycloid of two cusps, one at Y_1 and the other at the diametrically opposite point. This is our first evolute and P_1C_1 is its normal.



If $\rho_1 = P_1P_2$ is the radius of curvature of the first evolute, then $\rho_1 = (1/2)d\rho_0/d\gamma = 3(\cos\gamma)/4$, $C_1P_2 = (1/2)\,P_1C_1$, and P_2 is the projection upon this line of C_2 the middle point of OC_1. Now we prove as above for P_1, that the locus of P_2, our second evolute, is a two-cusped epicycloid, this time traced by rolling the circle of diameter C_2C_1 upon the fixed circle of radius OC_2, and having its cusps at the point where the latter circle cuts the x-axis and at the diametrically opposite point.

The same reasoning may be repeated again and again, giving us for the nth evolute an epicycloid with cusps on the x-axis when n is even and on the y-axis when n is odd. The equations in the former case are $x_n = (1/2^{n+1})(3\cos\gamma - \cos 3\gamma)$, $y_n = (1/2^{n+1})(3\sin\gamma - \sin 3\gamma)$, while the minus signs are changed to plus signs for the latter case.

The length of an arc of any evolute after the first measured from the nearest cusp of the preceding evolute is equal to the radius of curvature of the preceding evolute. Now $\rho_{n-1} = 3(\sin\gamma)/2^n$ or $3(\cos\gamma)/2^n$, neglecting signs; hence the complete length of the nth evolute can be obtained from one or the other of these expressions by putting the sine or cosine equal to 1 and multiplying by 4. That is, it is $3/2^{n-2}$ (it is 6 for $n = 1$).

For an evolute whose cusps lie on the y-axis the element of area generated by that portion of the radius of curvature which lies outside of the corresponding fixed circle is equal to twice the element xdy for the circle. For the first evolute, for example, it is $(1/2)\cos^2\gamma d\gamma$. A similar relation, the axes being interchanged, holds for an evolute whose cusps lie on the x-axis. Therefore the area of any evolute of the system is 3 times the area of the corresponding circle; namely, for the nth evolute it is $3\pi/2^{2n}$.

Since the lengths and areas form geometrical progressions it is easy to find their sums.

NOTES AND NEWS.

It is hoped that readers of the Monthly will coöperate in contributing to the general interest of this department by sending items to H. P. Manning, Brown University, Providence, R. I.

Charles Leonard Bouton died at Cambridge, Mass., February 20, 1922. He was born at St. Louis, April 25, 1869. He graduated at the Washington University with the degree of M.Sc. in 1891, took the degree of A.M. at